Section 4-5 Locating Zeroes of a Polynomial Function

Even with the Rational Root Theorem and Descartes Rule of Signs, the location of some roots is not possible. When roots are real but irrational, these techniques fail to locate the missing roots.

The Location Principle helps to locate these irrational, real roots:

Location Principle - Suppose $y = f(x)$ represents a polynomial function with real coefficients. If a and b are two numbers with $f(a)$ negative and $f(b)$ positive, the function has at least one real zero between a and b.

"If $f(a) > 0$ and $f(b) < 0$, then the function also has at least one real zero between a and b."

Ex. $f(x) = x^3 - 4x^2 - 2x + 8$

3 roots
2 or 0 (+) real roots
1 (-) real root
$\frac{p}{q} = \pm 8, \pm 4, \pm 2, \pm 1$

By inspection, using +8 is a large (+) #. using (-8) is a large (-) #. Check integer values between -4 and +4.

r	1	-4	-2	8	
-4	1	-8	30	-112	
-3	1	-7	19	-49	
-2	1	-6	10	-12	Sign change (root between -2 & -1)
-1	1	-5	3	5	
0	1	-4	-2	8	
1	1	-3	-5	3	Sign change (root between 1 & 2)
2	1	-2	-6	-4	
3	1	-1	-5	-7	
4	1	0	-2	0	← root

Notice that the outcome gave us 1(−) real root between −2 & −1 and 2(+) real root with 1 between (1 & 2) and the second at 4.

Now, try to narrow in on the two roots found with the Location Principle.

$f(-1.5) = -1.4$; $f(-1.4) = 0.2$

The first root is ≈ -1.4

$f(1.5) = -0.6$; $f(1.6) = -1.3$; $f(1.4) = 0.1$

The second root is ≈ 1.4

The roots for $P(x)$ are $-1.4, 1.4, 4$.

Ex: $f(x) = x^3 - 3x + 1$

3 roots

2(+) or 0(+) real roots

1(−) real root

$\frac{p}{q} = \pm 1$

check roots between −3 and +3

r	1	0	−3	1
−3	1	−3	6	−17
−2	1	−2	1	−1
−1	1	−1	−2	3
0	1	0	−3	1
1	1	1	−2	−1
2	1	2	1	3
3	1	3	6	17

Sign change (root −2 → −1)

Sign change (root 0 → 1)

Sign change (root 1 → 2)

check $f(-1.5) = 2.1$
$f(-1.4) = 2.5$
$f(-1.6) = 1.7$
$f(-1.8) = 0.6$
$f(-1.9) = -0.2$

$x \approx -1.9$ $\quad x \approx 1.5$

$x \approx 0.4$

Check $f(0.5) = -0.4$
$f(0.4) = -0.1$

$f(1.5) = -0.1$

check $f(1.3) = -0.7$

The roots for $P(x) = x^3 - 3x + 1$ are approximately at $-1.9, 0.4,$ and 1.5

While the Location Principle helps with finding irrational real roots, there still is a need to try and find the boundaries of possible values to be checked for roots or sign changes.

Upper Bound Theorem - For c as a positive number and P(x) divided by $x-c$, if the quotient and remainder have no change in sign, then P(x) has no real zero greater than c

From the previous problem we see that for $x-2$ the quotient and remainder have the coefficients of 1, 2, 1, 3. Notice that all of the numbers are the same sign, (+). This tells us that $x=2$ is our upper bound of possible roots.

A lower bound is an integer less than or equal to the least real zero. The lower bound is found by using the upper bound technique for $P(-x)$.

Lower Bound Theorem - If c is an upper bound of the zeroes of $P(-x)$, then $-c$ is a lower bound of the zeroes of $P(x)$.

From the previous problem, $f(-x) = -x^3 + 3x + 1$

$$\begin{array}{r|rrrr} & -1 & 0 & 3 & 1 \\ 4 & & -4 & -16 & -52 \\ \hline & -1 & -4 & -13 & -51 \end{array}$$

Note all of the terms are (-). This suggest that -4 is our lower bound. This means that there are no real roots less than -4. Remember that -3 did not have the same sign result.

Ex: $f(x) = x^4 - 3x^3 - 2x^2 + 3x - 5$

4 real roots

3 real (+) roots

or

1 real (+) roots

$\frac{p}{q} = \pm 5, \pm 1$

	1	-3	-2	3	-5	
-5	1	-8	38	-187	930	
-4	1	-7	26	-101	399	
-3	1	-6	16	-45	130	
-2	1	-5	8	-13	21	sign change (root location)
-1	1	-4	2	1	-6	
0	1	-3	-2	3	-5	
1	1	-2	-4	-1	-6	
2	1	-1	-4	-5	-15	
3	1	0	-2	-3	-14	sign change (root location)
4	1	1	2	11	39	→ upper Bound
5	1	2	8	43	210	

$P(-x) = x^4 + 3x^3 - 2x^2 - 3x - 5$

$$\begin{array}{r|rrrrr} 6 & 1 & 3 & -2 & -3 & -5 \\ & & 6 & 54 & 312 & 1854 \\ \hline & 1 & 9 & 52 & 309 & 1849 \end{array}$$

-6 is the lower bound

Summary:

$f(-1.2) = -4.2$

$f(-1.5) = -1.8$ ✓

$f(-1.6) = 3.9$

$f(3.2) = -9.3$

$f(3.5) = 2.4$ ✓

roots known: $\approx -1.5,\ 3.5$

lower Bound -6

upper Bound 4