Section 4-4 The Rational Root Theorem

For any polynomial, $P(x)$, the factors of the lead coefficien[t] and the final constant will contribute to possible real roots of $P(x)$.

Rational Root Theorem - Let $a_n x^n + a_{n-1} x^{n-1} + \ldots a_1 x + a_0 = 0$ represent a polynomial equation of degree n with integral coefficients. If a rational number $\frac{p}{q}$, where p and q have no common factors, is a root of the equation, then p is a factor of a_0 and q is a factor of a_n.

This theorem helps to identify the possible rational roots for $P(x)$.

Ex. $6x^3 + 11x^2 - 3x - 2 = 0$

$\uparrow q$ (6) $\uparrow p$ (−2)

factors of p: $\pm 2, \pm 1$

factors of q: $\pm 6, \pm 3, \pm 2, \pm 1$

The rational root possibilities include:

$$\pm \frac{1}{3}, \pm \frac{1}{6}, \pm \frac{2}{3}, \pm 1, \pm \frac{1}{2}, \pm 2$$

Using synthetic division, we can test these possible rational real roots to locate solutions for $P(x)$.

Integral Root Theorem-

Let $P(x) = x^n + a_{n-1}x^{n-1} + \dots a_1 x + a_0 = 0$, we see that the lead coefficient is 1, each other coefficient is an integer and $a_0 \neq 0$. This results in the rational roots of this equation must be integral factors of a_0.

If the factors of P are integers and $q = 1$ then $\frac{P}{q}$ = the factors of P.

Descartes Rule of Signs - This rule is used to determine the nature of the roots of $P(x)$.

For $P(x)$ arranged in descending order, the number of positive real roots will equal to the number of sign changes in $P(x)$ or an even multiple of this number.

For $P(-x)$ arranged in descending order, the number of negative real roots will equal to the number of sign changes in $P(-x)$ or an even multiple of this number.

Ex. $P(x) = 2x^5 + 3x^4 - 6x^3 + 6x^2 - 8x + 3$

$+ \quad + \quad - \quad + \quad - \quad +$

no yes yes yes yes

4 (+) real roots
or
2 (+) real roots
or
0 (+) real roots

$P(-x) = -2x^5 + 3x^4 + 6x^3 + 6x^2 + 8x + 3$

$- \quad + \quad + \quad + \quad + \quad +$

yes no No No No

1 (−) real root

$P(x)$ has 5 roots.

Possible outcomes:
- 4 (+) real, 1 (−) real
- 2 (+) real, 1 (−) real, 2 imaginary
- 0 (+) real, 1 (−) real, 4 imaginary

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#10. $x^3 + 2x^2 - 5x - 6 = 0$

$P = -6$; $Q = 1$; $\frac{p}{q} = \pm 6, \pm 3, \pm 2, \pm 1$

$$\begin{array}{r|rrrr} & 1 & 2 & -5 & -6 \\ 2 & & 2 & 8 & 6 \\ \hline & 1 & 4 & 3 & 0 \\ -1 & & -1 & -3 & \\ \hline & 1 & 3 & 0 & \\ -3 & & -3 & & \\ \hline & 1 & 0 & & \end{array}$$

$x = 2, -1, -3$

#12. $x^4 - 5x^3 + 9x^2 - 7x + 2 = 0$

$p = 2$; $q = 1$; $\frac{p}{q} = \pm 2, \pm 1$

$$\begin{array}{r|rrrrr} 1 & 1 & -5 & 9 & -7 & 2 \\ & & 1 & -4 & 5 & -2 \\ \hline 2 & 1 & -4 & 5 & -2 & 0 \\ & & 2 & -4 & 2 & \\ \hline & 1 & -2 & 1 & 0 & \end{array} \longrightarrow$$

$1x^2 - 2x + 1 = 0$

$(x-1)(x-1) = 0$

$x = 1, 1$

roots $x = 1, 2$

$x = 1$ repeats 3 times

18. $f(x) = x^3 - 2x^2 - 8x$

yes no

1 (+) real root

1 (−) real root

0

$f(-x) = -x^3 - 2x^2 + 8x$

no yes

$f(x) = x(x^2 - 2x - 8) = x(x-4)(x+2)$

$\downarrow \quad \downarrow \quad \downarrow$

$0 \quad 4 \quad -2$

$x = 0, -2, 4$

21. $f(x) = x^4 + 2x^3 - 9x^2 - 2x + 8$

no yes no yes

2 (+) real roots
or
0 (+) real roots

$f(-x) = x^4 - 2x^3 - 9x^2 + 2x + 8$

yes no yes no

2 (−) real roots
or
0 (−) real roots

$p = 8$

$q = 1$; $\frac{p}{q} = \pm 8, \pm 4, \pm 2, \pm 1$

	1	2	−9	−2	8
2		2	8	−2	−8
	1	4	−1	−4	0
1		1	5	4	
	1	5	4	0	

$\longrightarrow \quad x^2 + 5x + 4 = (x+4)(x+1)$

$\downarrow \quad \downarrow$

$-4 \quad -1$

$x = 2, 1, -4, -1$