Section 4-3 The Remainder and Factor Theorems

Consider a function $f(x) = x^3 + 3x^2 - 5x + 7$ and a divisor of $x - 1$

$$
\begin{array}{r|l}
 & x^2 + 4x - 1 \\
\hline
x - 1 & x^3 + 3x^2 - 5x + 7 \\
 & -(x^3 - x^2) \\
\hline
 & 4x^2 - 5x + 7 \\
 & -(4x^2 - 4x) \\
\hline
 & -x + 7 \\
 & -(-x + 1) \\
\hline
 & 6
\end{array}
$$

Notice that the remainder from the division is 6.
If we consider the same function evaluated at $x = 1$, $f(1)$, we will get a value of 6.

$$f(1) = 1^3 + 3(1)^2 - 5(1) + 7 = 1 + 3 - 5 + 7 = 6$$

This outcome leads to the Remainder Theorem.

If a polynomial $P(x)$ is divided by $x - r$, the remainder is a constant $P(r)$, and $P(x) = (x - r) \cdot Q(x) + P(r)$ where $Q(x)$ is a polynomial with degree one less than the degree of $P(x)$.

r is the value to be evaluated in $P(x)$.
$Q(r)$ the resulting quotient
$P(r)$ the remainder from the division.

The division of a divisor into $P(x)$ can become tedious and time consuming. This problem is overcome by using the technique called <u>synthetic division</u>.

Technique to utilize synthetic division.

step 1 Arrange terms of $P(x)$ in descending order and include 0's for any missing value within $P(x)$.

step 2 Write out the divisor (r) on the shelf adjacent to the row of coefficients from step 1.

step 3 Bring down the first coefficient, multiply it by the divisor and place it below the second term of the row of coefficients and add.

step 4 Repeat step 3 until the last coefficient has been added.

step 5 The remaining number in the last position will be the remainder, $P(r)$. It will also be the function value resulting from $P(x)$. If the remainder is a 0, the value x that was evaluated in the polynomial will be a solution, (a root), of the function.

This leads to the Factor Theorem.

Factor Theorem - The binomial $x - r$ is a factor of the polynomial $P(x)$ if and only if $P(r) = 0$.

Example

17. $(x^4 - 8x^2 + 16) \div (x+2)$

$(x-r) = (x-(-2))$
$r = -2$

step 1

$$\begin{array}{r|ccccc} -2 & 1 & 0 & -8 & 0 & 16 \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ & x^4 & x^3 & x^2 & x & \text{constant} \end{array}$$

step 2

$$\begin{array}{ccccc} 1 & 0 & -8 & 0 & 16 \end{array}$$

step3

$$\begin{array}{r|ccccc} -2 & & -2 & & & \\ \hline & 1 & -2 & & & \end{array}$$

step4

$$\begin{array}{r|ccccc} -2 & 1 & 0 & -8 & 0 & 16 \\ & & -2 & 4 & 8 & -16 \\ \hline & 1 & -2 & -4 & 8 & 0 \end{array}$$

step5 The final value is 0. This indicates that -2 is a root of the polynomial, $P(x)$

The divisor = -2

$Q(x) = 1x^3 - 2x^2 - 4x + 8$

The remainder $P(r) = 0$

#23. $(x^3 - x + 6) \div (x-2)$

2	1	0	-1	6
		2	4	6
	1	2	3	12

divisor = 2

$Q(x) = 1x^2 + 2x + 3$

Remainder $P(r) = 12$

#33. $x^3 + 4x^2 - x - 4$

1	1	4	-1	-4
		1	5	4
	1	5	4	0

with $Q(x) = x^2 + 5x + 4$

This is factorable.

$x^2 + 5x + 4 = (x+4)(x+1)$

$x = -4, -1$

Final solution: $x = 1, -1, -4$

#29 $x^3 + x^2 - 4x - 4$

2	1	1	-4	-4
		2	6	4
	1	3	2	0

$Q(x) = x^2 + 3x + 2$

This is factorable.

$(x+2)(x+1) = 0$

$x = -2, -1$

Final solution: $x = 2, -2, -1$