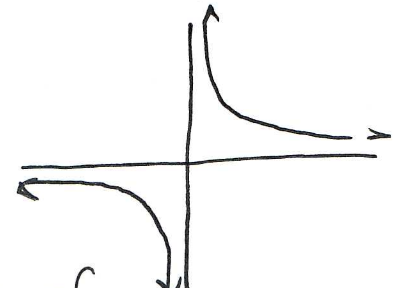
3-7. Graphs of Rational Functions

Rational functions, $f(x) = \frac{1}{x}$, take the form of $f(x) = \frac{g(x)}{h(x)}$ where $h(x) \neq 0$ and $g(x)$ and $h(x)$ are polynomials. For this type of function, the discontinuity is infinite. Because of this, the function approaches lines that act as boundaries. These lines are called asymptotes.

Ex. $y = \frac{1}{x}$



When we look for values of y when we let x approach 0 from the left, the y value goes to $-\infty$ and when we approach $x=0$ from the right, the y value goes to $+\infty$. So the graph never reaches $x=0$. This makes the y-axis $(x=0)$ a vertical asymptote.

Also we see that the x-axis $(y=0)$ is a boundary so we call it a horizontal asymptote.

This leads to a statement about vertical & horizontal asymptotes.

vertical $\rightarrow$ The line $x=a$ is a vertical asymptote if $f(x) \rightarrow \infty$ or $f(x) \rightarrow -\infty$ as $x \rightarrow a$ from the right or left.

horizontal $\rightarrow$ The line $y=b$ is a horizontal asymptote if $f(x) \rightarrow b$ as $x \rightarrow \infty$ or $-\infty$.

Solving for the asymptotes of a rational function.

① Look at the denominator, $h(x)$ an determine x-valu of any that could make the denominator become 0. This x-value will be a vertical asymptote.

② To solve for horizontal asymptotes, rework the function to solve it in terms of x. The resulting equation will lead to y values that create 0 values in the denominator. This will be the horizontal asymptote.

Ex. $f(x) = \dfrac{2x-1}{x+2}$

① Denominator $x+2$
If $x=-2$ then we get 0 in denominator. Therefo
$x=-2$ is a vertical asymp

② Solve function in terms of x.

$$y = \frac{2x-1}{x+2}$$

$$y(x+2) = 2x-1$$

$$xy+2y = 2x-1$$

$$2y+1 = 2x - xy$$

$$2y+1 = x(2-y)$$

$$\frac{2y+1}{2-y} = x$$

Asymptotes:
vertical $x=-2$
horizontal $y=2$

Note that when $y=2$, the equation is 0. Therefore $y=2$ is a horizontal asymptote.

An alternative method of determining the horizontal asymptote can be used when the degree of the denominator is greater than or equal to the degree of the numerator. This method uses the idea of limits as x is allowed to go to $+\infty$.

Ex. $y = \dfrac{3x+1}{x-3}$ $y = \dfrac{\frac{3x}{x} + \frac{1}{x}}{\frac{x}{x} - \frac{3}{x}}$ we divided each term by the degr of x in the denominator.

$y = \dfrac{3 + \frac{1}{x}}{1 - \frac{3}{x}}$ when $x \to +\infty$, $\frac{1}{x}$ & $-\frac{3}{x}$ will go to and will be cancelled, leaving only $y=3$. This equation is your horizontal asymptote.

ex. $y = \dfrac{2x^2 - x - 3}{x^2 + x + 1}$; $y = \dfrac{\frac{2x^2}{x^2} - \frac{x}{x^2} - \frac{3}{x^2}}{\frac{x^2}{x^2} + \frac{x}{x^2} + \frac{1}{x^2}}$

$y = \dfrac{2 - \frac{1}{x} - \frac{3}{x^2}}{1 + \frac{1}{x} + \frac{1}{x^2}}$; $y = \dfrac{2-0-0}{1+0+0} = 2$

$y=2$ is a horizontal asymptote.

A special type of asymptote occurs when the degree of the numerator is exactly one greater than the denominator.

Ex. $f(x) = \dfrac{3x^2+2x+1}{x+1}$

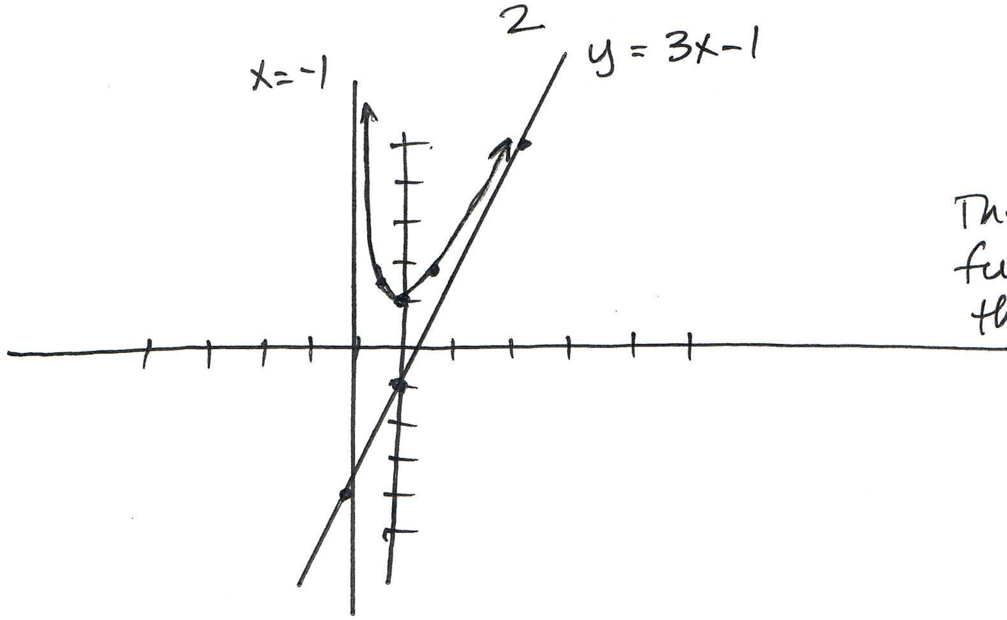
The degree of the numerator =
The degree of the denominator =

When this occurs, the asymptote is not vertical or horizontal but instead is slanted.

To solve for the equation of the slanted asymptote we simply need to divide the denominator into the numerator. The resulting answer without including the remainder will be the equation of the slant asymptote.

$$
\begin{array}{r l}
 & 3x - 1 \qquad R=2 \\
x+1 \,\big) & 3x^2+2x+1 \\
 & -(3x^2+3x) \\
\hline
 & -x+1 \\
 & -(-x-1) \\
\hline
 & 2
\end{array}
$$

The equation $y = 3x-1$ is the equation of the slant asymptote.



The graph of the function lies between the two asymptotes.

Example #2

Find the asymptotes.

$x = -3$ is the vertical asymptote

$$y = \frac{2}{x+3} + 1$$

$$y - 1 = \frac{2}{x+3}$$

$$(y-1)(x+3) = \frac{2}{x+3} \cdot (x+3)$$

$$xy + 3y - x - 3 = 2$$

$$xy - x = 2 - 3y + 3$$

$$x(y-1) = 5 - 3y$$

$$x = \frac{5-3y}{y-1}$$

This equation tells us that $y = 1$ is our horizontal asymptote.

Practice Problem - solve for the asymptotes.

$$y = \frac{3x+1}{x-3}$$