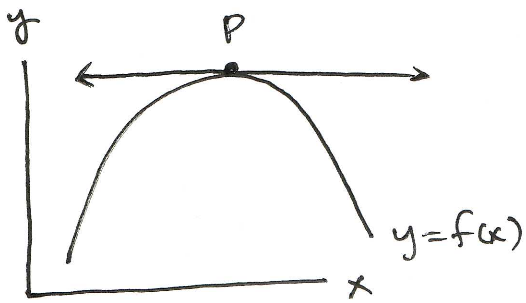
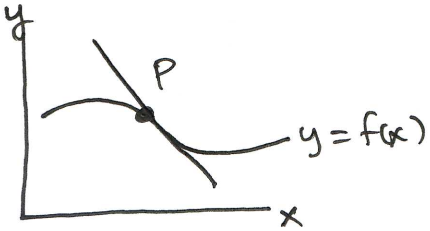
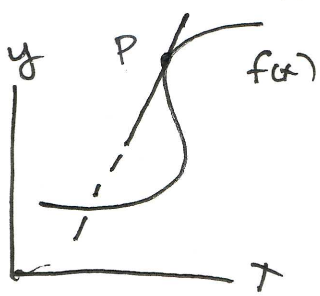
The Derivative and the Tangent Line Problem



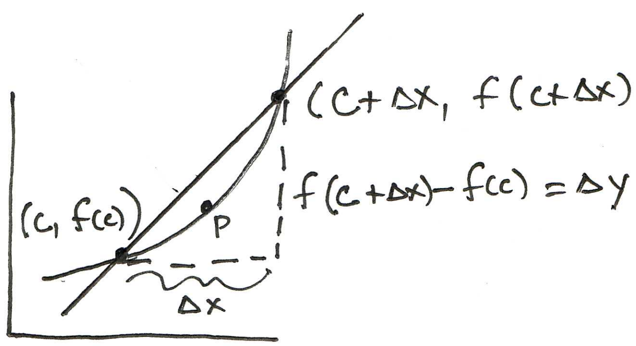
For this f(x), the tangent line at P provide's a slope at P.



For this f(x), a tangent line at P intersects f(x) which causes difficulty as tangents do not cross function lines.



For this f(x), the tangent line at P is effective at P, but intersects the function twice, causing difficuety as tangents intersect at one Point.



The slope of the secant line leads to an acceptable technique for solving for slope of the tangent line at P.

$$m = \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

with an increasingly smaller value for Δx, we approach a limit that results in the slope of the tangent at P.

* Definition of a Tangent Line with slope m.

If f is defined on an open interval containing c, and if $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = m$ the limit exists, then the line passing through $(c, f(c))$ with slope m is the tangent line to the graph of f at the point $(c, f(c))$.

Ex. Slope of a Linear Function

$f(x) = 2x - 3$ at $P(2,1)$ $c = 2$

$$\lim_{\Delta x \to 0} \frac{f(2+\Delta x) - f(2)}{\Delta x} = \lim_{\Delta x \to 0} \left[\frac{(2(2+\Delta x)-3) - (2(2)-3)}{\Delta x}\right] =$$

$$\lim_{\Delta x \to 0} \frac{4 + 2\Delta x - 3 - 4 + 3}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 2 = 2$$

* For linear equations, the limit definition of slope of f agrees with the definition of the slope of a line. The slope of linear function is constant for any point.

Ex Slope of a Nonlinear Function

$f(x) = x^2 + 1$ at $P_1(0,1)$ and $P_2(-1,2)$

Let $(c, f(c))$ represent an arbitrary point on the graph f.

$$\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = \lim_{\Delta x \to 0} \left[\frac{((c+\Delta x)^2 + 1) - (c^2+1)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \frac{c^2 + 2c(\Delta x) + (\Delta x)^2 + 1 - c^2 - 1}{\Delta x} = \lim_{\Delta x \to 0} \frac{2c(\Delta x) + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(2c + \Delta x)}{\Delta x} = \lim_{\Delta x \to 0} 2c + \Delta x = 2c$$

So the slope at <u>any</u> point $(c, f(c))$ on the graph of f is $m = 2c$. At $P_1(0,1)$ $m = 2(0) = 0$ and at $P_2(-1,2)$ $m = 2(-1) = -2$

This process works for all non vertical tangent lines.

The Derivative of a Function

The limit used to define the slope of a tangent line is also used to define one of the two fundamental operations of calculus - differentiation.

Definition of the Derivative of a Function

The derivative of f at x is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Provided the limit exists. For all x for which the limit exists, $f'(x)$ is a function of x.

Common Notations for the Derivative

$$f'(x), \quad \frac{dy}{dx}, \quad \frac{d(f(x))}{dx}, \quad D_x[y]$$

The notation $\frac{dy}{dx}$, read as "the derivative of y with respect to x".

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= f'(x)$$

Example: Finding the Derivative by the Limit Process

$f(x) = x^3 + 2x$: determine $f'(x)$

$$f'(x) = \lim_{\Delta x \to 0} \left(\frac{f(x+\Delta x) - f(x)}{\Delta x} \right) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 + 2(x+\Delta x) - (x^3 + 2x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2x + 2\Delta x - x^3 - 2x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x \left(3x^2 + 3x\Delta x + (\Delta x)^2 + 2\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(3x^2 + 3x\Delta x + (\Delta x)^2 + 2\right) = 3x^2 + 2$$

Example: $f(x) = \sqrt{x}$ determine $f'(x)$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{(\sqrt{x+\Delta x} - \sqrt{x})}{\Delta x} \left(\frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}} \right)$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x) - x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \text{ or } \frac{1}{2}(x)^{-1/2}$$

Example: $f(x) = \frac{2}{x}$ Determine $f'(x)$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\frac{2}{x+\Delta x} - \frac{2}{x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{x}{x}\left(\frac{2}{x+\Delta x}\right) - \left(\frac{x+\Delta x}{x+\Delta x}\right)\left(\frac{2}{x}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{2x - 2x - 2\Delta x}{x(x+\Delta x)}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{-2\Delta x}{x(x+\Delta x)}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-2\Delta x}{x(x+\Delta x)(\Delta x)} = \lim_{\Delta x \to 0} \frac{-2}{x(x+\Delta x)} = -\frac{2}{x^2} \text{ or } -2x^{-2}$$

The Constant Rule

The derivative of a constant function is 0. If c is a real number, then

$$\frac{d}{dx}(c) = 0$$

Proof: Let $f(x) = c$. $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{c - c}{\Delta x} = 0.$$

$\therefore$ The slope of a horizontal line is 0.

The Power Rule

If n is a rational number, then the function $f(x) = x^n$ is differentiable and $f'(x) = n x^{n-1}$

Recall the general binomial expansion rule

$$(x+\Delta x)^n = x^n + n x^{n-1}(\Delta x) + \frac{n(n-1)x^{n-2}}{2}(\Delta x)^2 + \dots + (\Delta x)^n$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^n + n x^{n-1}(\Delta x) + \frac{n(n-1)x^{n-2}}{2}(\Delta x)^2 + \dots (\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[n x^{n-1} + \frac{n(n-1)x^{n-2}}{2}(\Delta x) + \dots + (\Delta x)^{n-1} \right]$$

$$= \lim_{\Delta x \to 0} n x^{n-1} + 0 + 0 + \dots + 0 = n x^{n-1}$$

special case $\quad f(x) = x \quad$ then $\quad f'(x) = 1x^{1-1} = 1x^0 = 1$

Examples:

$f(x) = x^3 \qquad f'(x) = 3x^2$

$g(x) = \sqrt[3]{x} = x^{1/3} \qquad g'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$

$y = \frac{1}{x^2} = x^{-2} \qquad \frac{dy}{dx} = -2x^{-3} = \frac{-2}{x^3}$

The Constant Multiple Rule

If f is a differentiable function and c is a real number, then $c \cdot f$ is also differentiable and

$$\frac{d\,(c \cdot f(x))}{dx} = c \cdot f'(x)$$

Proof: $$\lim_{\Delta x \to 0} \frac{c \cdot f(x+\Delta x) - c \cdot f(x)}{\Delta x} = \lim_{\Delta x \to 0} c \left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= c \cdot f'(x)$$

Factor out the coefficient constant, take the derivative of $f(x)$ and then multiply c with the derivative.

The Sum and Difference Rules

The sum or difference of two differentiable functions is differentiable and is the sum or difference of their derivatives.

$$\frac{d\,(f(x) + g(x))}{dx} = f'(x) + g'(x)$$

$$\frac{d\,(f(x) - g(x))}{dx} = f'(x) - g(x)$$

Proof: $\dfrac{d\left(f(x)+g(x)\right)}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{\left[f(x+\Delta x) + g(x+\Delta x) - f(x) - g(x)\right]}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x} + \dfrac{g(x+\Delta x) - g(x)}{\Delta x}$

$= f'(x) + g'(x)$

Example:

$f(x) = x^3 - 4x + 5$ the functions of x^3, $-4x$, and 5 can be differentiate simultaneously and added together

$\dfrac{d(x^3)}{dx} = 3x^2, \quad \dfrac{d(-4x)}{dx} = -4, \quad \dfrac{d(5)}{dx} = 0$

So $f'(x) = 3x^2 - 4$

Example: $g(x) = -\dfrac{x^4}{2} + 3x^2 - 2x$

$g'(x) = \dfrac{-4x^3}{2} + 6x^1 - 2 = -2x^3 + 6x - 2$

Derivatives of Sine and Cosine Functions

$f(x) = \sin x$ then $\dfrac{d(\sin x)}{dx} = \cos x$

$f(x) = \cos x$ then $\dfrac{d(\cos x)}{dx} = -\sin x$

Proof: $$\frac{d(\sin x)}{dx} = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x - \sin x + \cos x \sin \Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin x(\cos \Delta x - 1) + \cos x \sin \Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-\sin x(1 - \cos \Delta x) + \cos x \sin \Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} -\sin x \frac{(1 - \cos \Delta x)}{\Delta x} + \lim_{\Delta x \to 0} \cos x \left(\frac{\sin \Delta x}{\Delta x}\right)$$

$$= \lim_{\Delta x \to 0} -\sin x(0) + \lim_{\Delta x \to 0} \cos x(1)$$

$$= \cos x$$