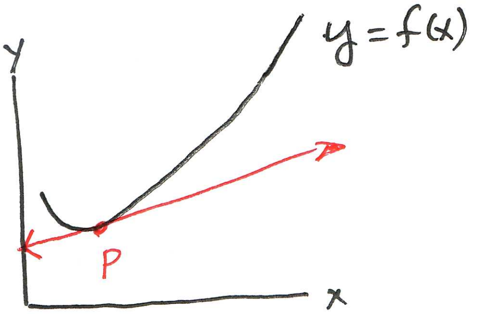
What is Calculus?

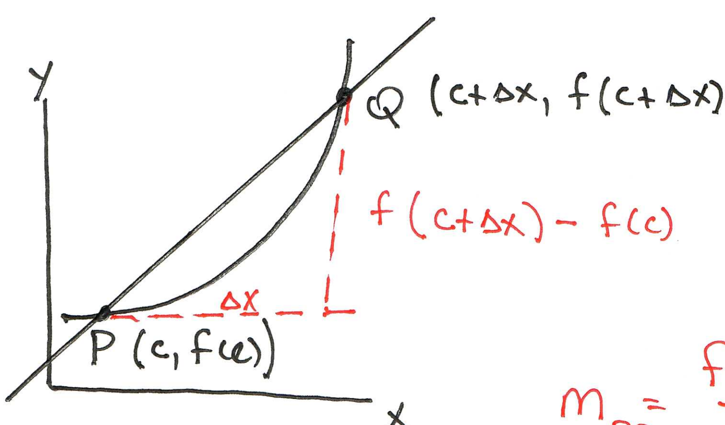
Calculus is the mathematics of change, tangent lines, slopes, areas, volumes, arc lengths, centroids, curvatures, and a variety of other concepts that enable people to model real-life situations.

The Tangent Line Problem



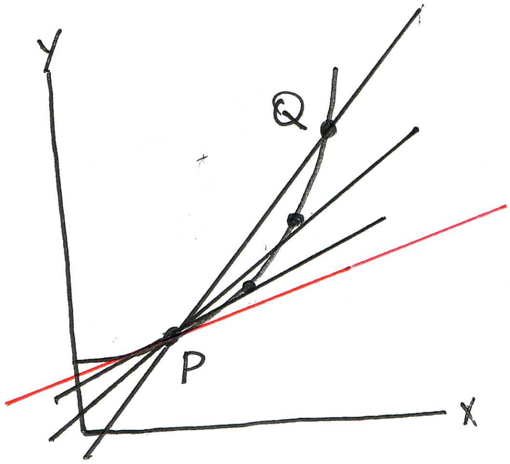
We investigate the slope along f(x) from both directions toward point P by use of limits.

Consider the following:



Secant line PQ where the horizontal change = Δx and the vertical change = $f(c+\Delta x) - f(c)$

$$m_{PQ} = \frac{f(c+\Delta x) - f(c)}{c+\Delta x - \Delta x} = \frac{f(c+\Delta x) - f(c)}{\Delta x}$$



With each successive secant line along the curve from Q toward P we approach the slope of the tangent line.

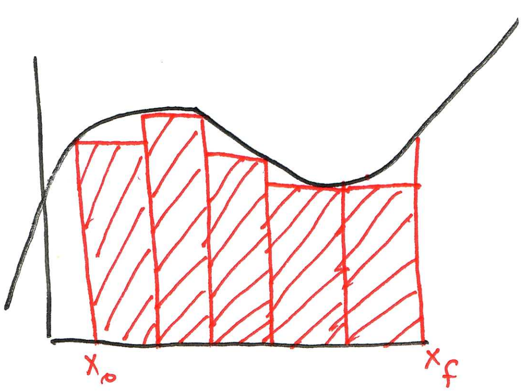
When such a "limiting position" exists the slope of the tangent line is said to be the limit of the slope of the secant line.

The slope of the secant line is continually reduced until it is transformed from a secant to a tangent line. The limit of the slope of the secant line approaches the slope of the tangent line.

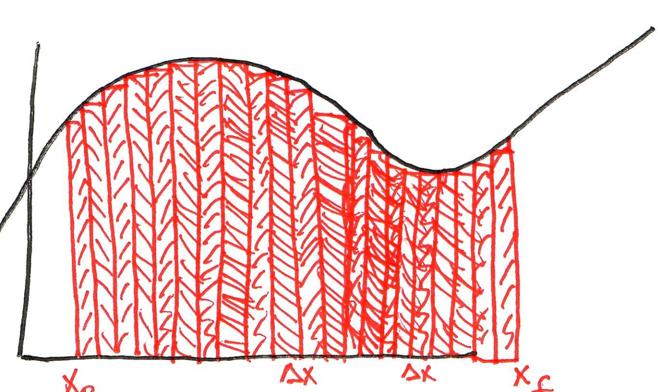
The Area Problem

Finding the area of a plane region that is bounded by the graphs of functions. Limits are also used to determine the area of the region.

The summation of the area of the rectangles below the curve gives an approximate value of the area. If we choose small rectangles Δx wide, the summation can give a very good measure of the area.

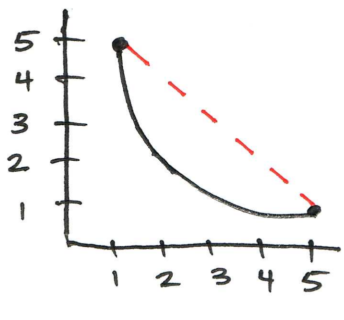


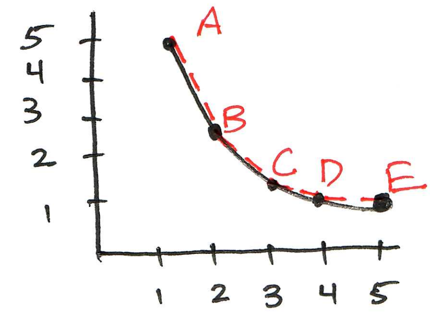
$$A = \sum_{x=x_o}^{x_f} \Delta x \cdot \sum_{y=y_o}^{y_f} \Delta y$$



Application

Consider the length of the graph of $f(x) = \frac{5}{x}$ from $(1,5)$ to $(5,1)$





a) Approximate the length of the curve by finding the distance between its two endpoints, as shown in the first figure.

b) Approximate the length of the curve by finding the lengths of the four line segments, as shown in the second figure.

c) Describe how you could continue this process to obtain a more accurate approximation of the length of the curve.

a) $(1,5) \to (5,1)$ $\quad d = \sqrt{(5-1)^2 + (1-5)^2} = \sqrt{32} = 5.66$

b)
AB $(1,5) \to (2,2.5)$ $\quad d = \sqrt{(2-1)^2 + (2.5-5)^2} = \sqrt{7.25} = 2.69$
BC $(2,2.5) \to (3,1.5)$ $\quad d = \sqrt{(3-2)^2 + (1.5-2.5)^2} = \sqrt{2} \approx 1.41$
CD $(3,1.5) \to (4,1.2)$ $\quad d = \sqrt{(4-3)^2 + (1.2-1.5)^2} = \sqrt{1.09} = 1.04$
DE $(4,1.2) \to (5,1)$ $\quad d = \sqrt{(5-4)^2 + (1-1.2)^2} = \sqrt{1.04} = 1.02$

6.16

c) By choosing smaller Δx values, we could approach the exact length of the graph from $(1,5)$ to $(5,1)$

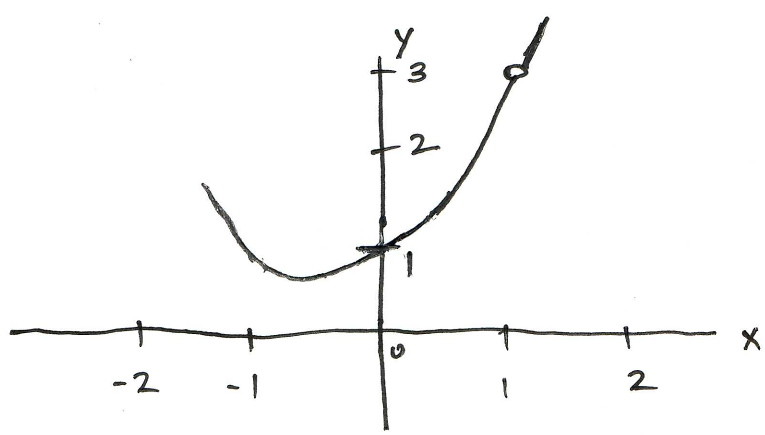
An Introduction to Limits

Consider $f(x) = \dfrac{x^3 - 1}{x - 1}$ $x \neq 1$

For all values other than $x = 1$, a curve can be sketched. However, at $x = 1$, the graph is uncertain. Approach $x = 1$ from both sides to isolate any existing pattern or outcome.

x	0.75	0.9	0.99	0.999	1	1.001	1.01	1.1	1.25
$f(x)$	2.313	2.710	2.970	2.997	3	3.003	3.030	3.310	3.813

$$\lim_{x \to 1} f(x) = 3$$



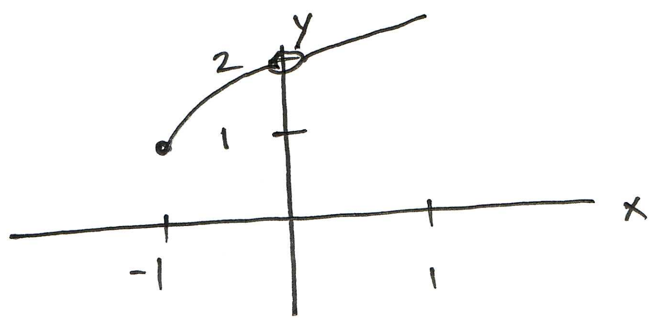
If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, the <u>limit</u> of $f(x)$ as x approaches c, is L.

$$\lim_{x \to c} f(x) = L$$

Example: Evaluate $f(x) = \dfrac{x}{(\sqrt{x+1} - 1)}$

$$\lim_{x \to 0} \frac{x}{(\sqrt{x+1} - 1)}$$

x	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01
f (x)	1.99499	1.99950	1.99995	2	2.00005	2.0005	2.00499



Note that the function is undefined at x=0 and yet f(x) appears to be approaching a limit as x approaches 0. The existence or nonexistence of f(x) at x=c has no bearing on the existence of the limit of f(x) as x approaches c.

Limits that Fail to Exist

Behavior that differs from the right and left.

$$\lim_{x \to 0} \frac{|x|}{x}$$

Consider the graph of

$$f(x) = \frac{|x|}{x}$$

for $x > 0$ $\quad \frac{|x|}{x} = 1$

for $x < 0$ $\quad \frac{|x|}{x} = -1$

This means that no matter how close x gets to 0, there will be both positive and negative x-values that yield $f(x) = 1$ and $f(x) = -1$. Specifically, if δ (delta) is a positive number, then for x-values satisfying the inequality $0 < |x| < \delta$, you can classify the values of $\frac{|x|}{x}$ as follows:

$(-\delta, 0)$

$-$ x-values yield $\frac{|x|}{x} = -1$

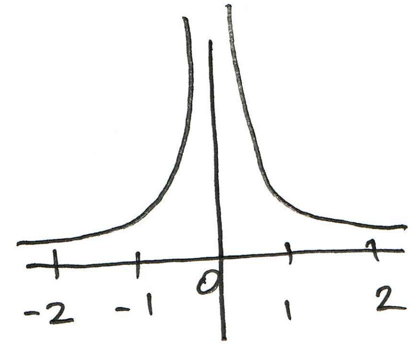
$(0, \delta)$

Positive x-values yield $\frac{|x|}{x} = 1$

This implies that the limit does not exist.

unbounded behavior

$$\lim_{x \to 0} \frac{1}{x^2}$$



As approaches 0 from either side, f(x) increases without bound. This means that by choosing x close enough to 0, you force f(x) to be as large as you want.

Because f(x) is not approaching a real number L as x approaches 0, you can conclude that the limit does not exist.

oscillating behavior

$$\lim_{x \to 0} \sin \frac{1}{x}$$

As x approaches 0, f(x) oscillates between -1 and 1.

Therefore, the limit does not exist because no matter how small you choose δ, it is possible to choose x_1 and x_2 within δ units of 0 such that $\sin\left(\frac{1}{x_1}\right) = 1$ and $\sin\left(\frac{1}{x_2}\right) = -1$

x	$\frac{2}{\pi}$	$\frac{2}{3\pi}$	$\frac{2}{5\pi}$	$\frac{2}{7\pi}$	$\frac{2}{9\pi}$	$\frac{2}{11\pi}$	$x \to 0$
$\sin \frac{1}{x}$	1	-1	1	-1	1	-1	Limit does not exist.

Common Types of Behavior Associated with the Nonexistence of a Limit

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side.

2. $f(x)$ increases or decreases without bound as x approaches c.

3. $f(x)$ oscillates between two fixed values as x approaches c.

A Formal Definition of a Limit

Developed by Augustin-Louis Cauchy

ε-δ (epsilon-delta) definition of a limit

Let ε represent a small positive number. Then the phrase "$f(x)$ becomes arbitrarily close to L" means that $f(x)$ lies in the interval $(L-\varepsilon, L+\varepsilon)$. Using absolute value, you write this as $|f(x)-L| < \varepsilon$.

Similarly, "x approaches c" means that there exists a number δ such that x lies in either the interval $(c-\delta, c)$ or the interval $(c, c+\delta)$. Expressed concisely $0 < |x-c| < \delta$.

The first inequality $0 < |x-c|$ expresses that $x \neq c$ and the difference between x and c is more than 0. The second inequality $|x-c| < \delta$ says that x is within a distance δ of c.

Definition of Limit

Let f be a function defined on an open interval containing c (except possibly at c) and let L be a real number.

$$\lim_{x \to c} f(x) = L$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x-c| < \delta$, then $|f(x) - L| < \epsilon$.

Therefore if $\lim_{x \to c} f(x) = L$ then the limits exists and the limit is L. Also, if the limit exists it is unique.

Example: Finding δ for a given ε.

$\lim_{x \to 3} (2x-5) = 1$ for δ such that $|(2x-5-1)| < 0.01$ whenever $0 < |x-3| < \delta$.

Solution: $\varepsilon = 0.01$ To find an appropriate δ, note that $|(2x-5)-1| = |2x-6| = 2|x-3|$.

Because the inequality $|(2x-5)-1| < 0.01$ is equivalent to $2|x-3| < 0.01$, you can choose $\delta = \frac{1}{2}(0.01) = 0.005$. This choice works because $0 < |x-3| < 0.005$ implies that

$|(2x-5)-1| = 2|x-3| < 2(0.005) = 0.01$

Finding a δ value for a given ϵ does not prove the existence of the limit. To do that you must prove that you can find a δ for any ϵ.

Example: Prove: $\lim_{x \to 2} (3x-2) = 4$

solution: show that for each $\epsilon > 0$, there exists a $\delta > 0$ such that $|(3x-2)-4| < \epsilon$ whenever $0 < |x-2| < \delta$.

Establish a connection between $|(3x-2)-4|$ and $|x-2|$

$|(3x-2)-4| = |3x-6| = 3|x-2|$ for $\epsilon > 0$, $\delta = \frac{\epsilon}{3}$

because $0 < |x-2| < \delta = \frac{\epsilon}{3}$ which implies that

$|(3x-2)-4| = 3|x-2| < 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$

Example

Given: $\lim_{x \to 2} x^2 = 4$

show that if $\epsilon > 0$, that this means $\delta > 0$ such that $|x^2 - 4| < \epsilon$ when $0 < |x-2| < \delta$.

To find an appropriate δ, begin by writing
$|x^2-4| = |(x+2)(x-2)| = |(x+2)||(x-2)|$.

For all x in the interval $(1, 3)$ (x values from 1 to 3 but not including 3)
$|x+2| < 5$.

So, letting δ be the minimum of $\frac{\epsilon}{5}$ and 1, it follows that whenever $0 < |x-2| < \delta$

$$|x^2-4| = |x-2||x+2| < \frac{\epsilon}{5}(5) = \epsilon$$

The ϵ-δ definition of a limit is used primarily to prove theorems about limits and to establish the existence or nonexistence of particular types of limits. For finding limits, you will learn techniques that are easier to use than the ϵ-δ definition of a limit.