Properties of Limits

Theorem 1.1 Basic Limits

	Example
1. $\lim_{x \to c} b = b$	$\lim_{x \to 2} 3 = 3$
2. $\lim_{x \to c} x = c$	$\lim_{x \to -4} x = -4$
3. $\lim_{x \to c} x^n = c^n$	$\lim_{x \to 2} x^2 = 2^2 = 4$

Theorem 1.2 Properties of Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

1. Scalar multiple $\quad \lim_{x \to c} [b \cdot f(x)] = bL$

2. Sum or difference $\quad \lim_{x \to c} [f(x) \pm g(x)] = L \pm K$

3. Product $\quad \lim_{x \to c} [f(x) \cdot g(x)] = L \cdot K$

4. Quotient $\quad \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K}$ provided $K \neq 0$

5. Power $\quad \lim [f(x)]^n = L^n$

Theorem 1.3 Limits of Polynomial and Rational Functions

If p is a polynomial function and c is a real number then $\lim_{x \to c} p(x) = p(c)$

If r is a rational function given by $r(x) = \frac{p(x)}{q(x)}$ and c is a real number such that $q(c) \neq 0$, then:

$$\lim_{x \to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

__Example__ $\lim_{x \to 2} (4x^2+3) = \lim_{x \to 2} 4x^2 + \lim_{x \to 2} 3$

$$= 4 \lim_{x \to 2} x^2 + \lim_{x \to 2} 3$$

$$= 4(2)^2 + 3 = 19$$

__Example__ $\lim_{x \to 1} \frac{x^2+x+2}{x+1}$

with the denominator not equal to 0 when $x=1$ we apply Th. 1.3.

$$\lim_{x \to 1} \frac{x^2+x+2}{x+1} = \frac{1^2+1+2}{1+1} = \frac{4}{2} = 2$$

Theorem 1.4 The Limit of a Function Involving a Radical

Let n be a positive integer. The following limit is valid for all c if n is odd and is valid for $c > 0$ if n is even.

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$$

Theorem 1.5 The Limit of a Composite Function

If f and g are functions such that $\lim_{x \to c} g(x) = L$ and $\lim_{x \to L} f(x) = f(L)$, then $\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right)$

$$= f(L)$$

Example If $\lim_{x \to 0} (x^2+4) = 0^2+4 = 4$ and $\lim_{x \to 4} \sqrt{x} = 2$

then $\lim_{x \to 0} \sqrt{x^2+4} = \sqrt{4} = 2$

Example 2 If $\lim_{x \to 3} (2x^2-10) = 2(3)^2-10 = 8$ and $\lim_{x \to 8} \sqrt[3]{x} = 2$

then $\lim_{x \to 3} \sqrt[3]{2x^2-10} = \sqrt[3]{8} = 2$

Theorem 1.6 Limits of Trigonometric Functions

Let c be a real number in the domain of the given trigonometric function.

1. $\lim\limits_{x \to c} \sin x = \sin c$
2. $\lim\limits_{x \to c} \cos x = \cos c$
3. $\lim\limits_{x \to c} \tan x = \tan c$
4. $\lim\limits_{x \to c} \cot x = \cot c$
5. $\lim\limits_{x \to c} \sec x = \sec c$
6. $\lim\limits_{x \to c} \csc x = \csc c$

Examples

$$\lim_{x \to 0} \tan x = \tan(0) = 0$$

$$\lim_{x \to \pi} x \cos x = \left(\lim_{x \to \pi} x\right)\left(\lim_{x \to \pi} \cos x\right) = (\pi)(\cos \pi) = -\pi$$

$$\lim_{x \to 0} \sin^2 x = \lim_{x \to 0} (\sin x)^2 = (0)^2 = 0$$

Theorem 1.7 Functions that Agree at All but one Point

Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$$

Example

$$\lim_{x \to 1} \frac{x^3-1}{x-1}$$

Let $f(x) = \frac{x^3-1}{x-1}$ then $\frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}} = x^2+x+1 = g(x) \quad x \neq 1$

$$\lim_{x \to 1} \frac{x^3-1}{x-1} = \lim_{x \to 1} \frac{(x-1)(x^2+x+1)}{(x-1)}$$

$$= \lim_{x \to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}}$$

$$= \lim_{x \to 1} (x^2+x+1) = \lim_{x \to 1} x^2 + \lim_{x \to 1} x + \lim_{x \to 1} 1$$

$$= 1^2 + 1 + 1 = 3$$

A strategy for Finding Limits

1. Learn to recognize which limits can be evaluated by direct substitution. (Theorems 1.1 – 1.6)
2. If the limit of $f(x)$ as x approaches c cannot be evaluated by direct substitution, try to find a function g that agrees with f for all x other than $x = c$. (Choose g such that the limit of $g(x)$ can be evaluated by direct substitution.)
3. Apply Theorem 1.7 to conclude analytically that
$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = g(c).$$
4. Use a graph or table to reinforce your conclusion.

Cancellation and Rationalization Techniques

Ex. $\lim_{x \to -3} \dfrac{x^2+x-6}{x+3}$

$\lim_{x \to -3} (x^2+x-6) = 0$ Direct substitution fails.

$\lim_{x \to -3} (x+3) = 0$

Instead try: $f(x) = \dfrac{x^2+x-6}{x+3} = \dfrac{(x+3)(x-2)}{(x+3)} = x-2 = g(x)$, $x \neq -3$

using Theorem 1.7 it follows that

$$\lim_{x \to -3} \frac{x^2+x-6}{x+3} = \lim_{x \to -3} (x-2) = -5$$

Direct substitution into $g(x)$ succeeds.

This is an example of Cancellation Technique.

Ex. $\lim_{x \to 0} \dfrac{\sqrt{x+1}-1}{x}$

$\lim_{x \to 0} \sqrt{x+1} - 1 = 0$ Direct substitution fails.

$\lim_{x \to 0} x = 0$

Try rewriting the numerator using rationalization.

$$\frac{\sqrt{x+1}-1}{x} = \left(\frac{\sqrt{x+1}-1}{x}\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right) \leftarrow \text{conjugate of numerator}$$

$$= \frac{(x+1)-1}{x(\sqrt{x+1}+1)} = \frac{x}{x(\sqrt{x+1}+1)} = \frac{1}{(\sqrt{x+1}+1)}, \quad x \neq 0$$

Now use Theorem 1.7

$$\lim_{x \to 0} \frac{\sqrt{x+1}-1}{x} = \lim_{x \to 0} \frac{1}{\sqrt{x+1}+1} = \frac{1}{1+1}$$

$$= \frac{1}{2}$$

using a Table of values that approach 0 from both sides reveals the limit is $\frac{1}{2}$.

The Squeeze Theorem

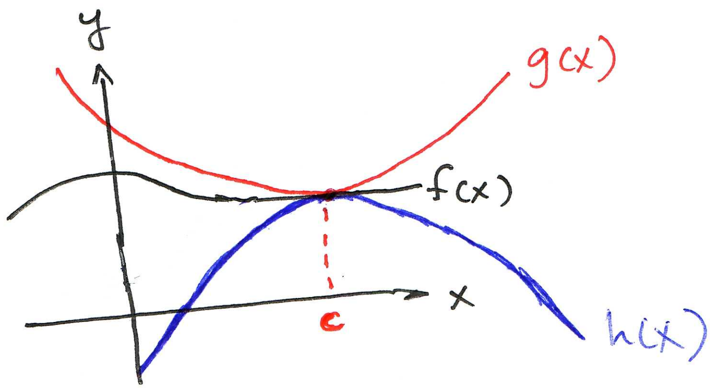
This is the limit of a function that is squeezed between two other functions.

Theorem 1.8 If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$$

then $\lim_{x \to c} f(x)$ exists and is also equal to L.

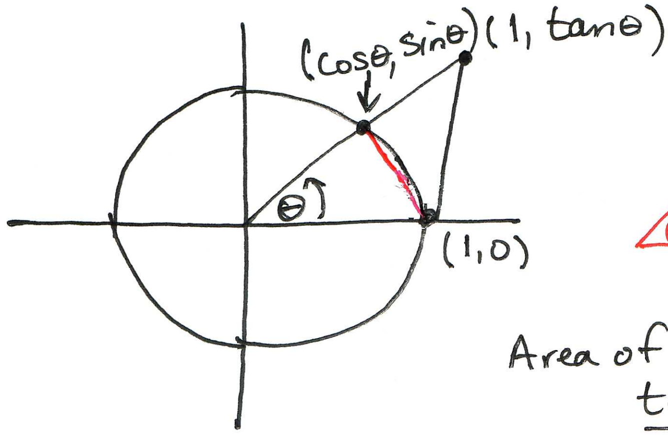
Illustration

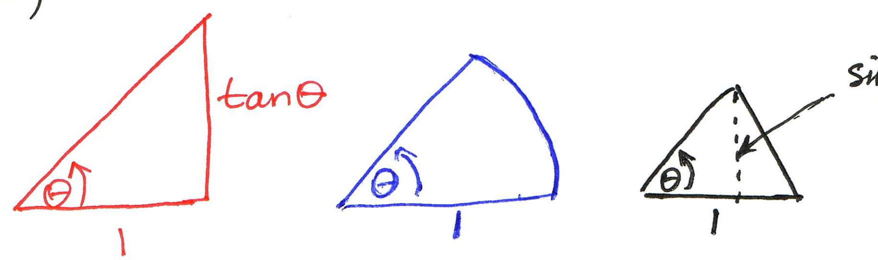


Theorem 1.9 Two Special Trigonometric Limits

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$ 2. $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

Proof of Theorem 1.9





Area of triangle $\geq$ Area of Sector $\geq$ Area of Δ

$$\frac{\tan\theta}{2} \geq \frac{\theta}{2} \geq \frac{\sin\theta}{2}$$

$$A_{\Delta_1} = \frac{1}{2}bh$$

$$A_{\Delta_1} = \frac{1}{2}(1)(\tan\theta)$$

$$\boxed{A_{\Delta_1} = \frac{\tan\theta}{2}}$$

$$A_{sector} = \frac{\theta}{2\pi} \cdot \pi r^2$$

$$A_{sector} = \frac{\theta}{2\pi} \cdot \pi(1)^2$$

$$\boxed{A_{sector} = \frac{\theta}{2}}$$

$$A_{\Delta_2} = \frac{1}{2}bh$$

$$= \frac{1}{2}(1)(\sin\theta)$$

$$\boxed{A_{\Delta_2} = \frac{\sin\theta}{2}}$$

multiply each expression by $\frac{2}{\sin\theta}$

gets you to:

$$\frac{1}{\cos\theta} \geq \frac{\theta}{\sin\theta} \geq 1$$

using reciprocals and reversing the inequalities yields

$$\frac{\cos\theta}{1} \leq \frac{\sin\theta}{\theta} \leq 1$$

Because $\cos\theta = \cos(-\theta)$ and $\frac{(\sin\theta)}{\theta} = \frac{\sin(-\theta)}{-\theta}$ we can conclude that this inequality is valid for all nonzero θ in the open interval $(-\pi/2, \pi/2)$. Finally, because $\lim_{\theta \to 0} \cos\theta = 1$ and $\lim_{\theta \to 0} 1 = 1$, you can apply the Squeeze Theorem to conclude that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

Example Find the limit:

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \left(\frac{\sin x}{x}\right)\left(\frac{1}{\cos x}\right)$$

since $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0} \frac{1}{\cos x} = 1$

we get $\lim_{x \to 0} \frac{\tan x}{x} = 1 \cdot 1 = 1$

Example Find the limit:

$$\lim_{x \to 0} \frac{\sin 4x}{x} = 4\left(\lim_{x \to 0} \frac{\sin 4x}{4x}\right).$$ Let $y = 4x$

and substitute we get $4\left(\lim_{y \to 0} \frac{\sin y}{y}\right) = 4(1) = 4$

Assignments: Pg 65 problems 5-22
Page 65 problems 23-36, 37-40
Page 66 Problems 49-56, 67-75, 83-88